Test 1 Computational Methods of Science
October 2022

Duration: 90 minutes

In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results.

Consider on (0,1) the differential equation

$$-\frac{d}{dx}(\cos(x)\frac{du}{dx}) + \frac{du}{dx} = \tan(x)$$

with boundary conditions $\frac{du}{dx}(0) - u(0) = 3$ and $u(1) = -2$.

1. [2.0] Derive the weak (Galerkin) form and the associated function space of this problem. Give also the bilinear and linear form, where in the bilinear form only first-order derivatives appear.

2. [2.0] Choose the remaining freedom in the bilinear form such that it becomes non-negative and show that it is actually positive definite then. Next show that it is also coercive. Is there an associated minimization problem? Explain your answer.

3. [2.0] Formulate the Lax-Milgram theorem and use it to show that the weak form derived in the previous parts is well-posed according to the Lax-Milgram theorem.

4. [1.0] Assume we approximate the solution on a subspace V_h spanned by the basis functions $\phi_0(x), \cdots, \phi_{n-1}(x)$, i.e. $u = \sum_{i=0}^{n-1} c_i\phi_i$. Give an expression for A_{ij} with A the matrix of the resulting linear system. Show that the matrix will be positive definite.

5. [2.0] Consider the space V_h of piecewise linear interpolation polynomials with interpolation points $x_i = ih$ with $h = 1/n$ and $v_h(1) = 0$ for all $v_h \in V_h$. Give the linear basis $\{\phi_0, \cdots, \phi_{n-1}\}$ for V_h. Using these linear basis functions, compute the diagonal elements A_{ii} for $i = 0, \cdots, n-1$ of the matrix A which occurs when approximating the solution on the space V_h.